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Shocks and energy dissipation in inviscid fluids: a question posed by Lord Rayleigh

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Lord Rayleigh argued that after a discontinuity develops in a one-dimensional compression wave in an ideal inviscid fluid some sort of motion must continue. Arguments are given in support of this view and a suggestion is made as to what that motion might be. The relationship of this motion to that proposed by Onsager for incompressible inviscid turbulent flows is discussed.

1. Introduction

In his famous paper 'On a Difficulty in the Theory of Sound' G. G. Stokes (1848), after showing that a discontinuity inevitably develops (where the characteristics meet) in a compression wave in an ideal, i.e. inviscid, fluid in which the pressure is proportional to the density, goes on to write:

> .... some motion or other will go on, and we might wish to know what the nature of that motion was.

These two phrases state the subject of this paper. The second expresses a point of view; the first seems self-evident, as it evidently was to Stokes. There is, of course, no present general interest in this problem because in real gases a shock wave is known to form in these circumstances.

Stokes proposed that the subsequent motion was a wave of finite amplitude and showed that such a motion is consistent with mass and momentum conservation:

$$\rho_1 u_1 = \rho_2 u_2, \tag{1}$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2. \tag{2}$$

In a letter to Stokes dated June 2, 1887 (reprinted in Truesdell's preface to Stokes's Mathematical and Physical Papers, 1966), Rayleigh pointed out that the proposed flow does not satisfy the energy equation:

$$\int_0^1 \frac{\mathrm{d}p}{\rho} + \frac{u_1^2}{2} = \int_0^2 \frac{\mathrm{d}p}{\rho} + \frac{u_2^2}{2}$$

or, with $p = c^2\rho$, and c a constant,

$$c^2 \ln \rho_1 + \frac{u_1^2}{2} = c^2 \ln \rho_2 + \frac{u_2^2}{2}. \tag{3}$$

Recognizing the validity of the objection and remarking that Sir William Thompson had earlier pointed out the error, Stokes removed the proposed solution when the paper was published in his collected works.

In the revised paper, Stokes writes that his mistake was not an unnatural one because in his model (to use current terminology) describing a reversible mechanical system, the conservation of energy is contained in the conservation of mass and momentum, an observation that raises the question: how can a solution to (1) and (2) fail to satisfy (3)? Rayleigh (1894) provides an answer by showing that if a force acts between station 1 and 2, first to the left and then to the right, such that the net force is zero, then (1) and (2) are satisfied, but energy is either added or withdrawn so that (3) is not satisfied.

2. Rayleigh's question

Having ruled out the discontinuous wave as the motion after the development of the discontinuity, Rayleigh still wished to know what the motion in an ideal fluid was, and argued incisively that the question was a valid one. He knew, of course, what the motion in a real viscous fluid was: he discussed the matter in, among other places, the paper in which he worked out the structure of the shock wave. In this paper (Rayleigh, 1910)† he wrote:

> When discontinuity sets in, a state of things exists to which the usual differential equations are inapplicable; and the subsequent progress of the motion has not been determined. It is probable, as suggested by Stokes, that some sort of reflection would ensue. In regard to this matter we must be careful to keep purely mathematical questions distinct from physical ones. We shall see later how the tendency to discontinuity may be held in check by forces of a dissipative character. But this has nothing directly to do with the mathematical problem of determining what would happen to waves of finite amplitude in a medium, free from viscosity, whose pressure is under all circumstances proportional to the density. To suppose that the problem has no solution would seem to be tantamount to admitting an inherent contradiction in the assumption, usually made in hydrodynamics, of a continuous fluid subject to Boyle's law. It would be strange if the necessity of a molecular constitution for gases could be established by such an argument.

Virtually the same statement appears in both the 1877 and 1894 editions of *The Theory of Sound*. The difficulty is present, of course, when $p/p_0 = (\rho/\rho_0)^\beta$ and β is any positive constant. We denote the constant as β not γ to repeat Rayleigh's emphasis that what is under discussion is an ideal continuum not a gas for which β would denote the ratio of specific heats.

In spite of the clarity of the above quotation, the question it raises bears repeating: what is the motion in an inviscid fluid in which $p/p_0 = (\rho/\rho_0)^\beta$ after a compression wave steepens to a discontinuity?

3. The Friedrichs discussion

The only writer, to the author's knowledge, to address this matter further is K. O. Friedrichs (1954), who quotes this passage, but rejects the idea that some motion must follow. Instead, he concludes that an ideal fluid cannot exist under these conditions. He writes:

> Not only is it true that gases as they actually occur in nature can no longer be described as ideal under these circumstances, but it is also *impossible in principle* that an ideal gas could exist under the circumstances. (Emphasis added)

† It is remarkable that this paper and G. I. Taylor's on the same subject (Taylor, 1910) were received by the Royal Society within three days of each other. Taylor does not comment on Rayleigh's worry about the ideal fluid; his paper concerns shocks in real gases.

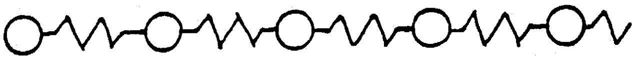

FIGURE 1. Spring-bead model.

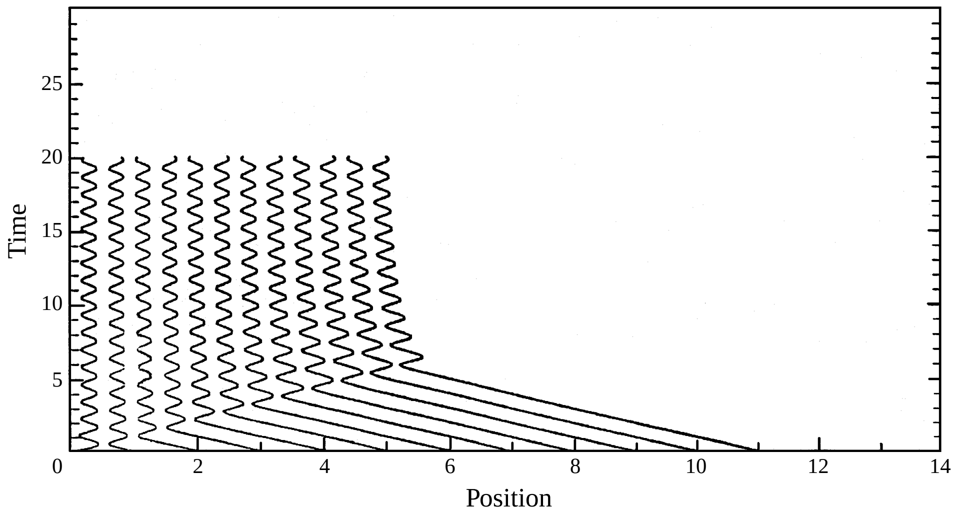

FIGURE 2. Bead motions in a shock flow.

'Impossible in principle' in this context would seem to mean inherently contradictory, to use Rayleigh's words, but Friedrichs does not point out any such contradiction. The only justification for this conclusion is, apparently, the non-existence of a solution to the flow problem. That Friedrichs is forced to such a conclusion is an indication of the degree of seriousness with which he considered Rayleigh's position.

Friedrichs closes his discussion of the matter with the following remark:

> This quotation [Rayleigh's above] is given to show that at times, though perhaps rarely, a natural conception about a physical situation does not lead to the right assumption about the existence of an associated mathematical problem.

It is proposed in the following that Rayleigh was right and Friedrichs wrong, i.e. that there is a mathematical problem associated with this physical situation.

4. The von Neumann model

A resolution to the dilemma was suggested to the author by results from a model for one-dimensional flow gas flow. J. von Neumann (1944) noted that if the space derivative in the inviscid one-dimensional equations of motion is replaced by its finite difference representation, the resulting equations describe, exactly, the motion of a string of particles or 'beads' connected by (nonlinear) springs as sketched in figure 1. When one end of such an array moving to the left is brought to rest, the resulting bead motions are as shown in figure 2, from a calculation by Darin Beigie (private communication). Von Neumann proposed that the vibrations set up behind the 'shock' represent the dissipated energy and that in this way this conservative, reversible system could be used to study flows containing shocks.†

In the model when the dissipated energy per unit length is constant, the bead

† Lax (1986) and Hou & Lax (1991) have express doubts about the idea.

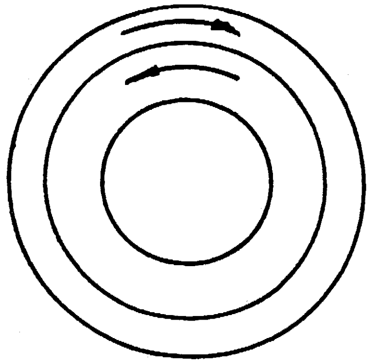

FIGURE 3. Inviscid flow in a torus.

vibration frequency increases inversely as the mesh spacing and the bead excursions fall correspondingly. Thus as the spacing and bead mass go to zero, the excursions go to zero and the frequency becomes infinite. It is proposed that this is what happens in the continuum. If this is the case, the steepening compression wave leads to regions of the flow in which derivatives do not exist, and the fluid properties in these regions will depend on the spatial variable like the Weierstrass function, i.e. continuously but with no derivative. (Such functions are discussed in Mandelbrot 1982). The conclusion to be drawn from this heuristic argument is that the downstream flow must be one without derivatives – though not necessarily one with only one-dimensional 'internal' motions; that could only be generated by exactly one-dimensional initial conditions.

5. Energy dissipation in turbulent flow

A closely related situation is discussed in a paper, Eyink (1994), brought to the author's attention by William C. Reynolds, on a conjecture of Onsager (1949) concerning energy dissipation in inviscid incompressible flows. The paper opens with a quotation from the Onsager reference which is as follows.

> It is of some interest to note that in principle, turbulent dissipation as described could take place just as readily without the final assistance by viscosity. In the absence of viscosity, the standard proof of the conservation of energy does not apply, because the velocity field does not remain differentiable!

Onsager remarks that to be applicable to flows under these conditions, the equations of motion have to be reformulated.

Eyink proves† the conjecture, and writes: 'We also construct a simple example of an initial velocity field [...] which has, in some sense, an instantaneous time-derivative of its energy which is non-vanishing. This suggests that its evolution forward in time provides a solution of the Euler equations which has a total energy which is either decreasing or increasing in time (by time-reversal invariance either behavior is possible.)'

To examine a situation analogous to that in the shock wave flow in which physically realizable initial conditions are imposed, consider the flow of an inviscid incompressible fluid in a torus as sketched in figure 3. Let the velocities in the inner and outer regions be such that the net angular momentum is zero. The instability of the vortex sheet separating the two regions leads to an intermingling of the different velocity

† Another proof is given in Constantin, E & Titi (1994).

fluids at all scales and consequently to a final state at rest in the mean. In this case the dissipated energy must reside in some sort of vortical motion, again of zero dimension. This motion, and that behind the shock wave, could well be called heat, or, more properly, internal energy. Rigorous solutions of the two problems would contain these thermodynamic terms.

6. Conclusion

Why was Rayleigh certain that there had to be a motion for all time in the shock wave problem, and why must the same be true for the turbulent motion in the torus? The answer proposed here is simply that the imagined fluids are *possible* real substances – meaning that they could exist without violating any physical law. Since such fluids must have motions for all time obeying mass and momentum conservation in response to physically realizable initial and boundary conditions (conditions given in these instances by the developing compression wave and by the unstable vortex sheet), and since these motions are described exactly by Euler equations, absent singularities, it was natural to expect solutions for all time. The difficulties arise because the conventional mathematical description of the motion is inapplicable in both circumstances. It is this aspect of the difficulty that is of interest here; Eyink discusses the possible relevance of the Onsager conjecture to turbulence theory.

There is, of course, a large literature on the fluid mechanics of inviscid fluids. The inviscid shock wave is an unsolved problem in this class of flows. If there were a solution in conventional mathematical terms it would be part of that classical literature, however unrealistic the solution might be. The inviscid, incompressible flow about an inclined flat plate with its singularities is an example of such a flow. The absence of a shock wave solution comes not from the unreality of the physical assumptions, but from the non-standard, in fluid mechanics, mathematics required for the solution. This problem is an especially clear example in classical mechanics of how a problem in physics may require a generalization in mathematics. Whether or not it would be useful to solve the problem may be questioned.

Although there is no current interest in the shock wave problem, the opposite is true of incompressible inviscid flow: the study of singularity formation and its implications is an active part of current turbulence research; see, for instance, Shelley (1992) and Eyink (1994). (With the vortex sheet in figure 3 replaced by a smooth profile, this flow might be an interesting example to consider.) It may be useful to note the similarities in the two flows discussed here. It is known that the energy dissipation rate in shock waves and in high-Reynolds-number turbulent flows is independent of the viscosity; shock wave thickness and turbulent velocity fields respond to viscosity changes so as to maintain the required dissipation rate. The discussion above suggests that this range of independence may include zero viscosity. For an incompressible fluid, the flow field, away from walls, in the limit of zero viscosity would appear to differ little from one of zero viscosity even though the Navier–Stokes equations are singular at the zero viscosity limit. The situation for the compressible fluid is more complex – to make a comparison, an equation of state of the inviscid fluid with the internal excitations would have to be derived.

These two inviscid fluids have another, related, similarity. Both obey the Second Law of thermodynamics, i.e. within closed, fixed boundaries, both would move from non-equilibrium initial states to equilibrium states where they would remain, presumably for infinite time, before returning to the initial states in accordance with the Poincaré theorem. This observation allows a choice between the two solutions mentioned by

Eyink in the quotation above. The equilibrium states arise solely from the nonlinearity of the equations of motion. This behaviour is remarkably similar to that of a gas, i.e. to a fluid composed of particles.

In addition to proposing an answer to Lord Rayleigh's question, the paper suggests that if the heuristic arguments presented can be made rigorous, then the Onsager conjecture (now proved) can be interpreted to say that the Euler equation solutions without derivatives can *conserve energy*, and that the motions of these inviscid fluids follow the Second Law.

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